Structure and Design of MMSE Channel Equalizers

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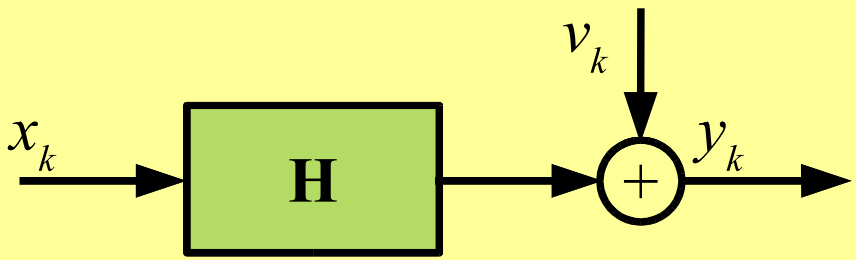
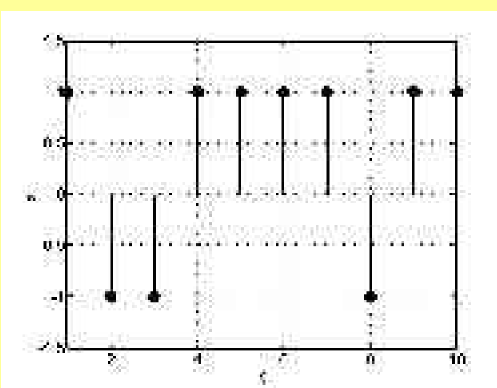
www.ee.hacettepe.edu.tr/~toker/equalizers.pdf

Intersymbol Interference (ISI)

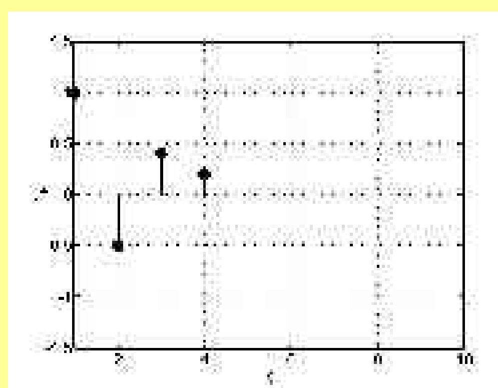


$$y_k = \sum_{n=0}^{N_H - 1} h_n x_{k-n} + v_k$$

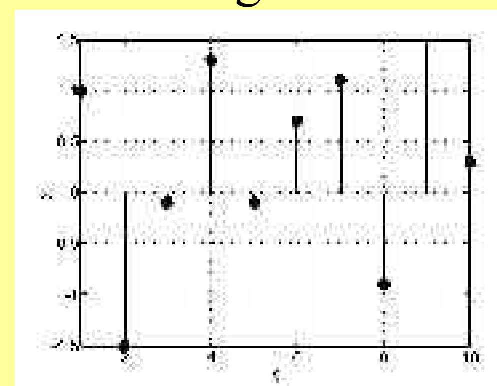
transmitted data



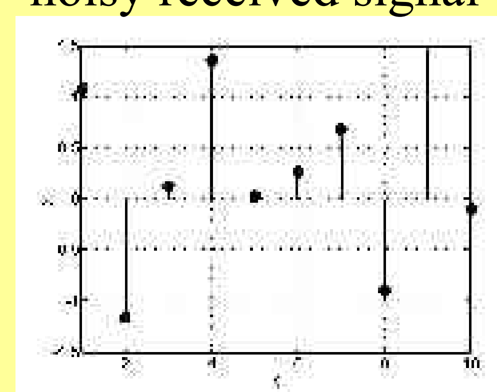
ISI channel



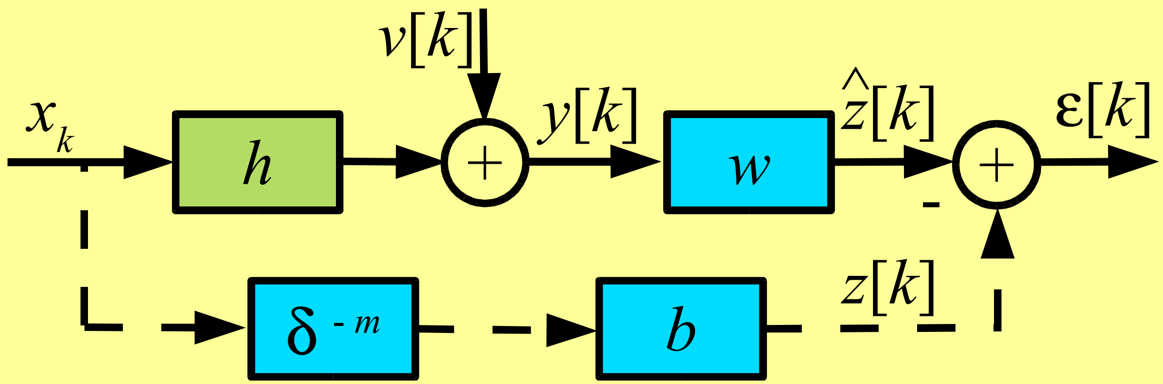
received signal-no noise



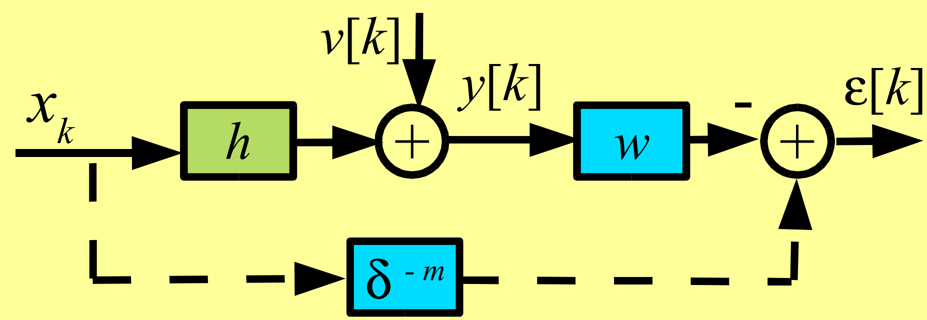
noisy received signal



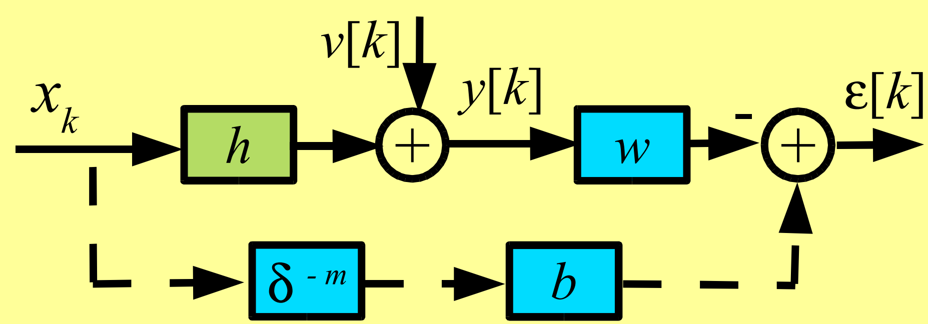
MMSE FIR Equalizers



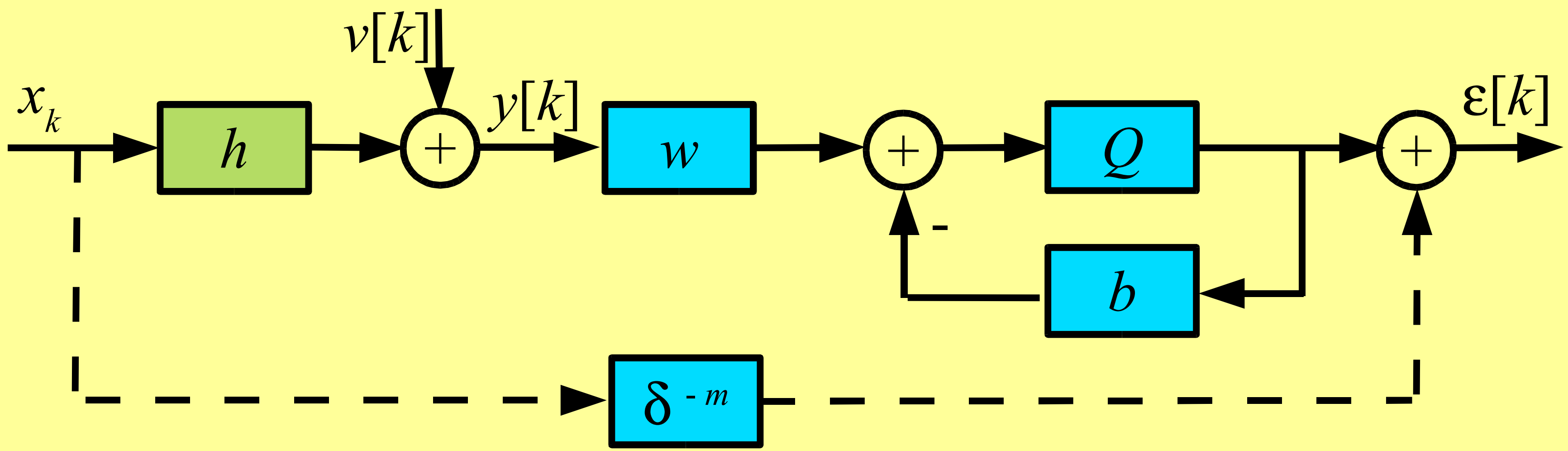
- *Equalizer*:
  $b = 1$



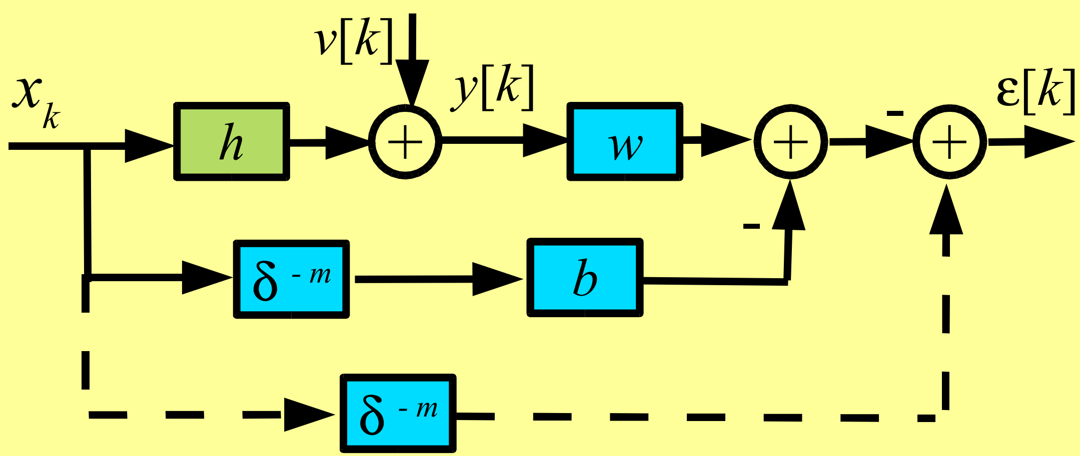
- <u>*Partial response equalizer*</u>:
  b: fixed (for example, $b=1+z^{-1}$)
- *Channel shortening equalizer*:
  b: design parameter



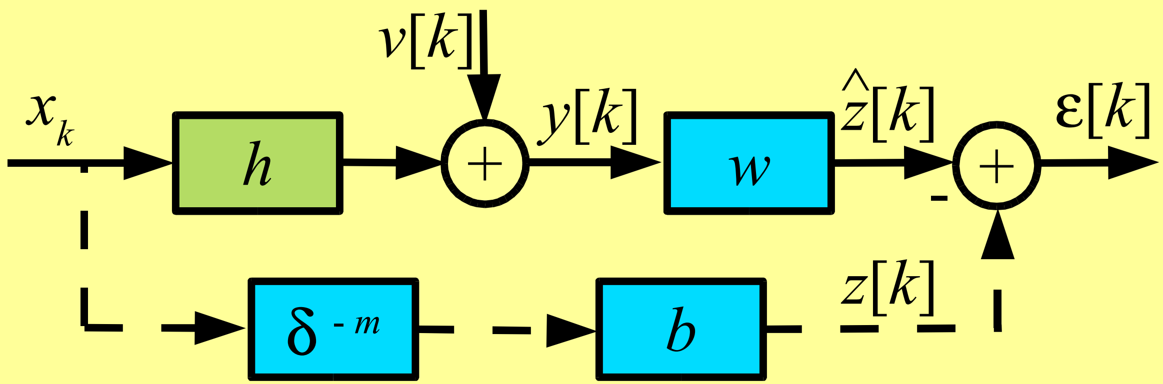
MMSE FIR
Decision Feedback Equalizer (DFE)



• Assumption: *Past decisions are correct*



MMSE FIR Equalizers



$$\hat{z}_k = \sum_{n=0}^{n_w-1} w_n^* y_{k-n} = \begin{bmatrix} w_0^* & w_1^* & \cdots & w_{n_w-1}^* \end{bmatrix} \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_{k-(n_{w-1})} \end{bmatrix} = \boldsymbol{w}^H \boldsymbol{y}_k$$

$$y_k = \sum_{m=0}^{n_w-1} w_m^* \left(\sum_{n=0}^{n_h-1} h_n x_{k-n-m} + v_{k-m} \right)$$

$$\hat{z}_k = \begin{bmatrix} w_0^* & w_1^* & \cdots & w_{n_w-1}^* \end{bmatrix} \left(\begin{bmatrix} h_0 & h_1 & \cdots & h_{n_h-1} & 0 & \cdots & 0 \\ 0 & h_0 & h_1 & \cdots & h_{n_h-1} & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & h_0 & h_1 & \cdots & h_{n_h-1} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \\ \cdots \\ x_{k-(n_h+n_w-2)} \end{bmatrix} + \begin{bmatrix} v_k \\ v_{k-1} \\ \vdots \\ v_{k-(n_{w-1})} \end{bmatrix} \right)$$

$$\hat{z}_k = \boldsymbol{w}^H \boldsymbol{H} \boldsymbol{x}_k + \boldsymbol{w}^H \boldsymbol{v}_k$$

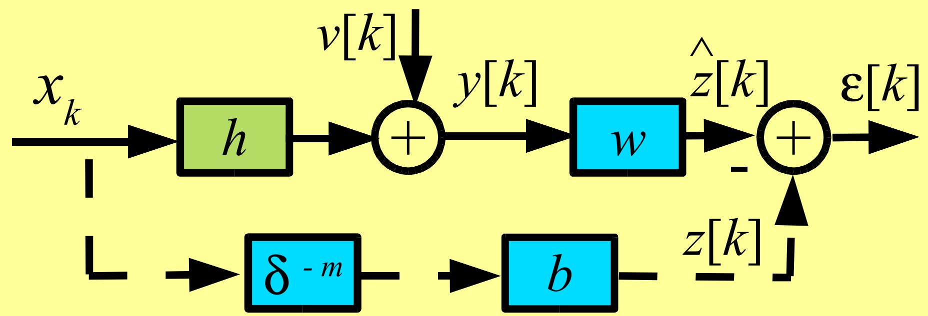
$$z_k = \sum_{n=0}^{n_b-1} b_{n-m}^* x_{k-n}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & b_0^* & b_1^* & \cdots & b_{n_b-1}^* & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-(n_w+n_h-2)} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{0}_{m\times 1} \\ \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}^H \boldsymbol{x}_k$$

$$z_k = \boldsymbol{b}^H \boldsymbol{x}_k$$

MMSE FIR Equalizers



- Error:

$$\varepsilon_k = z_k - \hat{z}_k = \left[\tilde{\boldsymbol{b}}^H - \boldsymbol{w}^H \boldsymbol{H}\right]\boldsymbol{x}_k - \boldsymbol{w}^H \boldsymbol{v}_k$$

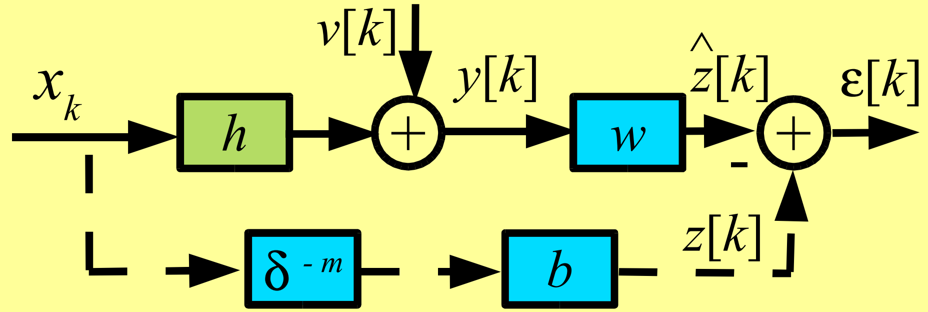
- Mean Square Error (MSE):

$$J = E\{\varepsilon_k \varepsilon_k^*\} = \left[\tilde{\boldsymbol{b}}^H - \boldsymbol{w}^H \boldsymbol{H}\right]\boldsymbol{R}_{xx}\left[\tilde{\boldsymbol{b}} - \boldsymbol{H}^H \boldsymbol{w}\right] + \boldsymbol{w}^H \boldsymbol{R}_{vv} \boldsymbol{w}$$

$$\boldsymbol{R}_{xx} = E\{\boldsymbol{x}_k \boldsymbol{x}_k^H\}$$

$$= E\left\{\begin{bmatrix} x_k \\ x_{k-1} \\ \cdots \\ x_{k-(n_h+n_w-2)} \end{bmatrix}\begin{bmatrix} x_k^* & x_{k-1}^* & \cdots & x_{k-(n_h+n_w-2)^*} \end{bmatrix}\right\} = \begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \cdots & r_{xx}[n_h+n_w-1] \\ r_{xx}^*[-1] & r_{xx}[0] & \cdots & r_{xx}[n_h+n_w-2] \\ \vdots & \ddots & \ddots & \vdots \\ r_{xx}^*[-(n_h+n_w-1)] & r_{xx}^*[-(n_h+n_w-2)] & \cdots & r_{xx}[0] \end{bmatrix}$$

MMSE FIR Equalizers



- J: quadratic function; optimum $\boldsymbol{w}$ is found by differentiating J wrt $\boldsymbol{w}$ and equating to zero[4] (or by orthogonality principle, $E\{\varepsilon[k]\boldsymbol{y}^H[k]\}=0$[1,2]).

$$\begin{aligned}\boldsymbol{w}^H &= \tilde{\boldsymbol{b}}^H \boldsymbol{R}_{xx}\boldsymbol{H}^H\left(\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^H+\boldsymbol{R}_{vv}\right)^{-1}\\ &= \tilde{\boldsymbol{b}}^H\left(\boldsymbol{R}_{xx}^{-1}+\boldsymbol{H}^H\boldsymbol{R}_{vv}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^H\boldsymbol{R}_{vv}^{-1} \quad (*)\end{aligned}$$

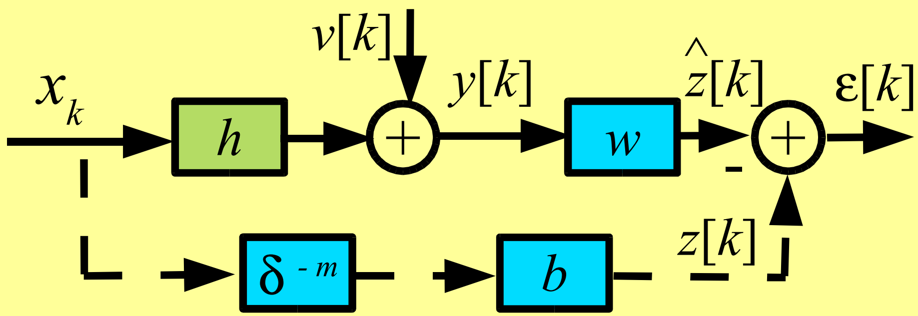
- <u>MMSE FIR equalizer</u>, $\boldsymbol{b}=1$: the optimum receiver is

$$\boldsymbol{w}^H=\boldsymbol{e}_m^H\left(\boldsymbol{R}_{xx}^{-1}+\boldsymbol{H}^H\boldsymbol{R}_{vv}^{-1}\boldsymbol{H}\right)^{-1}\underbrace{\boldsymbol{H}^H\boldsymbol{R}_{vv}^{-1}}_{\substack{\text{whitening}\\ \text{matched filter}}}$$

$$\boldsymbol{e}_m=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix} \leftarrow m$$

(*): Matrix Inversion Lemma: $\left(\boldsymbol{R}_{xx}^{-1}+\boldsymbol{H}^H\boldsymbol{R}_{vv}^{-1}\boldsymbol{H}\right)^{-1}=\boldsymbol{R}_{xx}-\boldsymbol{R}_{xx}\boldsymbol{H}^H\left(\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^H+\boldsymbol{R}_{vv}\right)^{-1}\boldsymbol{H}\boldsymbol{R}_{xx}$

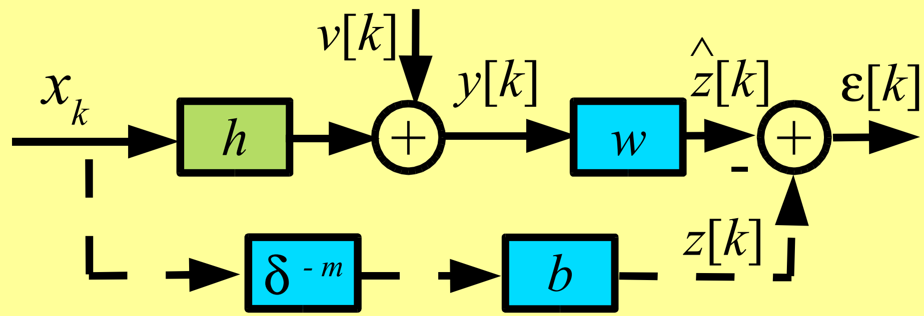
MMSE FIR Partial Response Equalizers



- <u>MMSE FIR fixed partial response equalizer</u>, ***b** fixed*:
- the optimum receiver is

$$\boldsymbol{w}^H = \tilde{\boldsymbol{b}}^H \left(\boldsymbol{R}_{xx}^{-1} + \boldsymbol{H}^H \boldsymbol{R}_{vv}^{-1} \boldsymbol{H} \right)^{-1} \underbrace{\boldsymbol{H}^H \boldsymbol{R}_{vv}^{-1}}_{\substack{whitening \\ matched\ filter}}$$

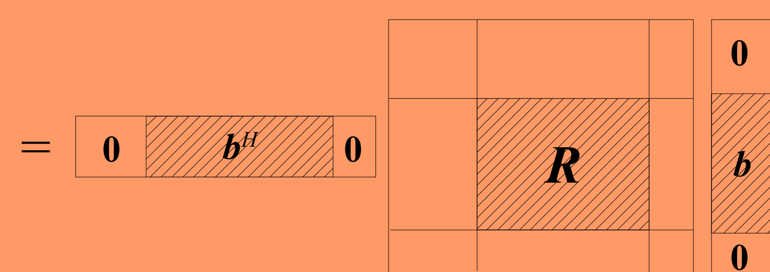
MMSE FIR Channel Shortening Equalizers



- MMSE FIR channel shortening equalizer, ***b*** *fixed length variable filter.*

- Substitute ***w*** into *J*:

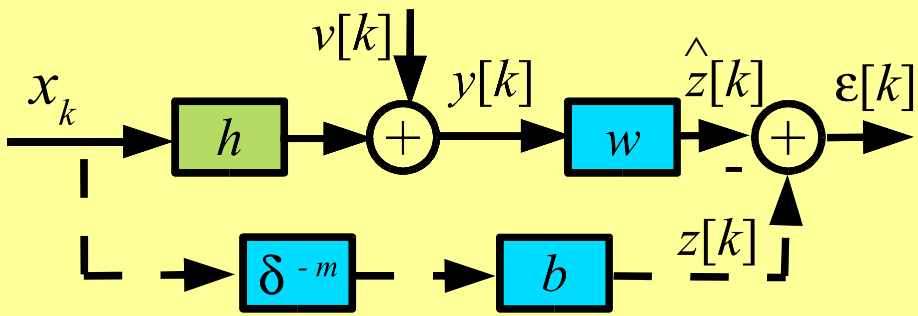
$$J = \tilde{\boldsymbol{b}}^H \left(\boldsymbol{R}_{xx}^{-1} + \boldsymbol{H}^H \boldsymbol{R}_{vv}^{-1} \boldsymbol{H} \right)^{-1} \tilde{\boldsymbol{b}}$$



$$= \boldsymbol{b}^H \boldsymbol{R} \boldsymbol{b}$$

- Impose a constraint on ***b*** in order to avoid the trivial case ***b***=0, *J=0*
  - *Orthogonality constraint* $\boldsymbol{b}^H\boldsymbol{b}=1$.

MMSE FIR Channel Shortening Equalizers



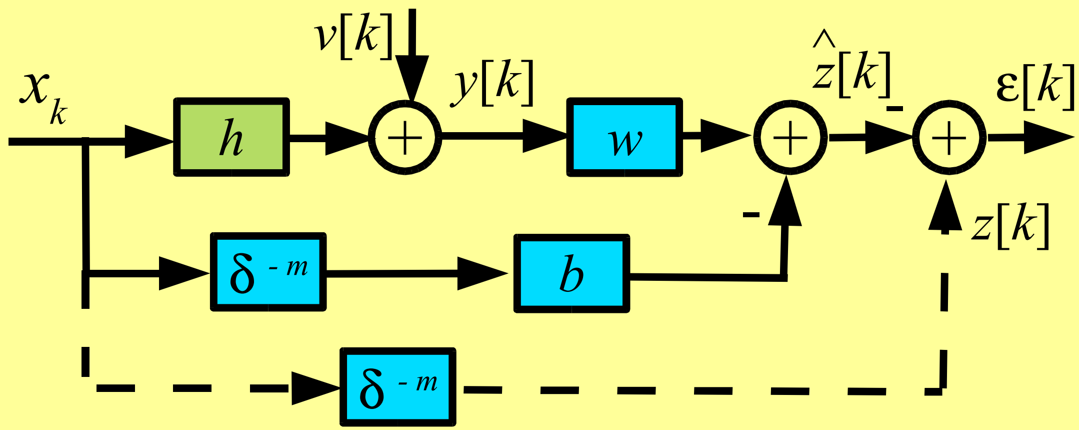
- Problem becomes :

$$
\begin{aligned}
&\textit{minimize} \quad J = \boldsymbol{b}^H \boldsymbol{R} \boldsymbol{b} \\
&\qquad = \boldsymbol{b}^H \boldsymbol{U} \Delta \boldsymbol{U}^H \boldsymbol{b} \\
&\qquad = \boldsymbol{b}^H \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_{n_b} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n_b} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^H \\ \boldsymbol{u}_2^H \\ \vdots \\ \boldsymbol{u}_{n_b}^H \end{bmatrix} \boldsymbol{b} \\
&\textit{subject to} \quad \boldsymbol{b}^H \boldsymbol{b} = 1
\end{aligned}
$$

- Constrained optimization: Solution for ***b*** is the eigenvector corresponding to the smallest eigenvalue, λ_{min}, of ***R***.

$$MMSE = \lambda_{min}$$

MMSE FIR Decision Feedback Equalizer (DFE)

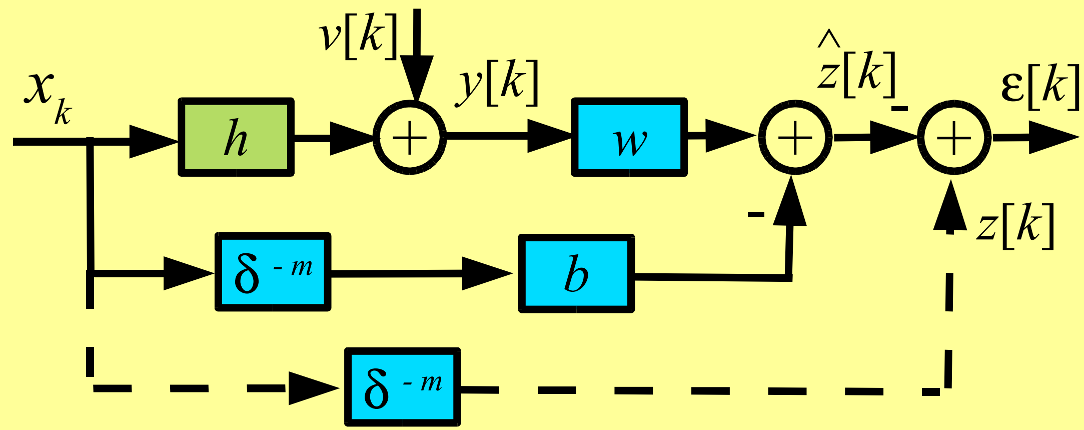


$$\hat{z}_k = \sum_{n=0}^{n_w-1} w_n y_{k-n} - \sum_{n=1}^{n_b-1} b_n x_{k-n-m} \quad \longleftarrow \quad y[k] = \sum_{n=0}^{n_h-1} h[n]\,x[k-n] + v[k]$$

$$= \sum_{n=0}^{n_h+n_w-1} c_n x_{k-n} - \sum_{n=1}^{n_b-1} b_n x_{k-n-m} + \sum_{n=0}^{n_w-1} w_n v_{k-n} \quad \longleftarrow \quad c[n] = w[n] * h[n]$$

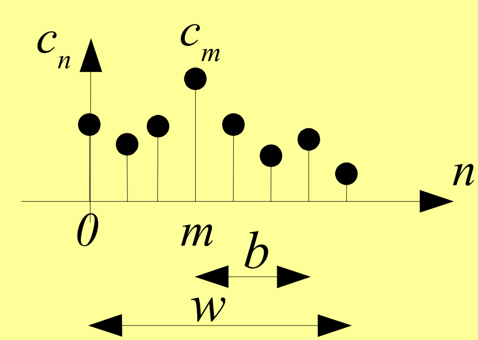
$$= \underbrace{c_m x_{k-m}}_{\text{information bearing cursor}} + \underbrace{\sum_{n=0}^{m-1} c_n x_{k-n}}_{\text{residual precursor ISI}} + \underbrace{\sum_{n=1}^{n_b-1} (c_{n+m} - d_n)\, x_{k-n-m}}_{\text{modeled postcursor ISI}} + \underbrace{\sum_{n=m+n_b+1}^{n_h+n_w-1} c_n x_{k-n}}_{\text{residual postcursor ISI}} + \underbrace{\sum_{n=0}^{n_w-1} w_n v_{k-n}}_{\text{filtered noise}}$$

MMSE FIR
Decision Feedback Equalizer (DFE)

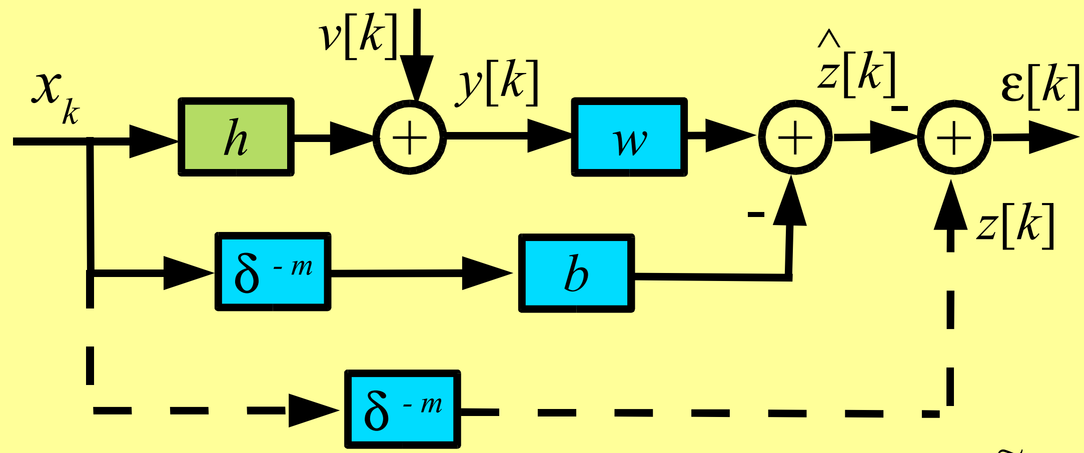


$$\hat{z}_k = \underbrace{c_m x_{k-m}}_{\substack{\text{information}\\ \text{bearing cursor}}} + \underbrace{\sum_{n=0}^{m-1} c_n x_{k-n}}_{\text{residual precursor ISI}} + \underbrace{\sum_{n=1}^{n_b-1} (c_{n+m} - b_n) x_{k-n-m}}_{\text{modeled postcursor ISI}} + \underbrace{\sum_{n=m+n_b+1}^{n_h+n_w-1} c_n x_{k-n}}_{\text{residual postcursor ISI}} + \underbrace{\sum_{n=0}^{n_w-1} w_n v_{k-n}}_{\text{filtered noise}}$$

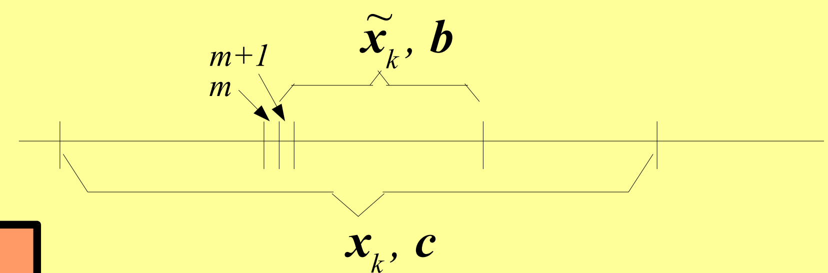
- Goals:
  - Feedforward filter:
    - shape $c_n = h_n * w_n$ so that $c_m \approx 1$
    - small residual ISI, $c_n \approx 0, n \neq m$
    - keep noise gain $\sum_k |f_k|^2$ as small as possible,
  - Feedback filter:
    - cancel the remaining ISI by matching $d_n = c_n + m$, $n = 1, \ldots, n_b - 1$



MMSE FIR
Decision Feedback Equalizer (DFE)



$$\varepsilon_k = x_{k-m} - \{\boldsymbol{w}^H(\boldsymbol{H}\boldsymbol{x}_k + \boldsymbol{v}_k) - \boldsymbol{b}^H\tilde{\boldsymbol{x}}_k\}$$



- Assumptions:

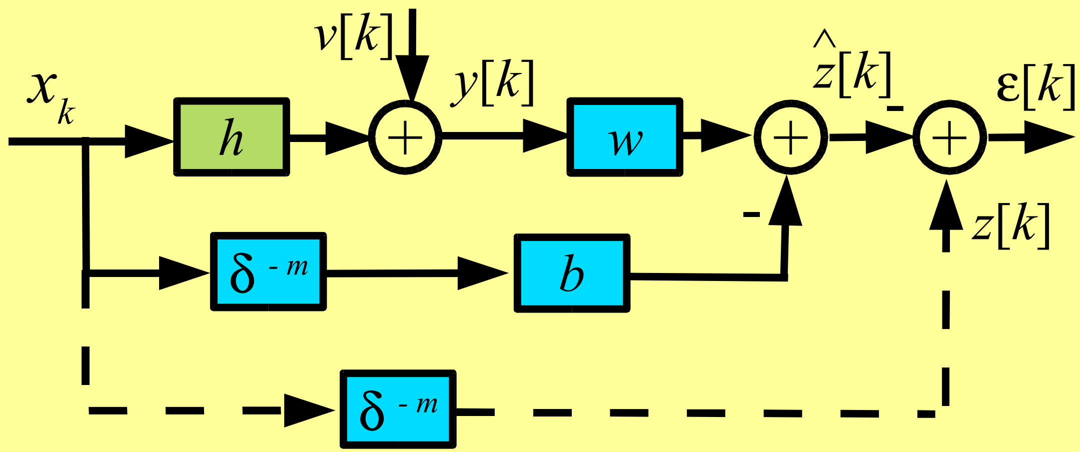
$E\{x_{k-m}\tilde{\boldsymbol{x}}_k\}=0,\ E\{x_{k-m}\boldsymbol{v}_k\}=\boldsymbol{0},\ E\{\boldsymbol{x}_k\boldsymbol{v}_k^H\}=\boldsymbol{0}$

$$\tilde{\boldsymbol{x}}_k = \boldsymbol{M}\boldsymbol{x}_k$$

$$\boldsymbol{M} = \begin{bmatrix}\boldsymbol{0}_{n_b\times m} & \boldsymbol{I}_{n_b\times n_b} & \boldsymbol{0}_{n_b\times(n_h+n_w-n_b-m)}\end{bmatrix}$$

$$\begin{aligned} J &= E\{\varepsilon[k]\varepsilon^*[k]\} \\ &= \sigma_x^2 - \left[\sigma_x^2\boldsymbol{e}_m^H + \boldsymbol{b}^H\boldsymbol{R}_{xx}\right]\boldsymbol{H}^H\boldsymbol{w} - \boldsymbol{w}^H\boldsymbol{H}\left[\sigma_x^2\boldsymbol{e}_m + \boldsymbol{R}_{xx}\boldsymbol{b}\right] + \boldsymbol{w}^H\left[\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^H + \boldsymbol{R}_{vv}\right]\boldsymbol{w} + \boldsymbol{b}^H\boldsymbol{R}_{xx}\boldsymbol{b} \end{aligned}$$

MMSE FIR Decision Feedback Equalizer (DFE)



$$J=\sigma_x^2-\left[\sigma_x^2\boldsymbol{e}_m^H+\boldsymbol{b}^H\boldsymbol{R}_{xx}\right]\boldsymbol{H}^H\boldsymbol{w}-\boldsymbol{w}^H\boldsymbol{H}\left[\sigma_x^2\boldsymbol{e}_m+\boldsymbol{R}_{xx}\boldsymbol{b}\right]+\boldsymbol{w}^H\left[\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^H+\boldsymbol{R}_{vv}\right]\boldsymbol{w}+\boldsymbol{b}^H\boldsymbol{R}_{xx}\boldsymbol{b}$$

- Optimum feedback filter, $\nabla_{\boldsymbol{b}} J=0$:

$$\boldsymbol{b}^H=\boldsymbol{w}^H\boldsymbol{H}\boldsymbol{M}^H$$

- Optimum feedforward filter, $J\leftarrow\boldsymbol{b},\ \nabla_{\boldsymbol{w}} J=0$:

$$\boldsymbol{w}^H=\sigma_x^2\boldsymbol{e}_m^H\boldsymbol{H}\left(\boldsymbol{H}\left(\boldsymbol{I}-\boldsymbol{M}^H\boldsymbol{R}_{xx}\boldsymbol{M}\right)\boldsymbol{H}^H+\boldsymbol{R}_{vv}\right)^{-1}$$

- Equivalent optimum feedforward filter, $\nabla_{\boldsymbol{w}} J=0$:

$$\boldsymbol{w}^H=\left(\sigma_x^2\boldsymbol{e}_m^H+\boldsymbol{b}^H\boldsymbol{M}\boldsymbol{R}_{xx}\right)\boldsymbol{H}^H\left(\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^H+\boldsymbol{R}_{vv}\right)^{-1}$$

Conclusions

- MMSE linear equalization is a well-studied field for combatting ISI channel.
- All MMSE equalizers share common feedforward filter structure:
  - MMSE equalizer: $\boldsymbol{w}^H = \boldsymbol{e}_m^H \boldsymbol{R}_{xx} \boldsymbol{H}^H \left(\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^H + \boldsymbol{R}_{vv} \right)^{-1}$
  - Partial response and channel shortening equalizers: $\boldsymbol{w}^H = \tilde{\boldsymbol{b}}^H \boldsymbol{R}_{xx} \boldsymbol{H}^H \left(\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^H + \boldsymbol{R}_{vv} \right)^{-1}$
  - Decision feedback equalizer: $\boldsymbol{w}^H = \left(\sigma_x^2 \boldsymbol{e}_m^H + \boldsymbol{b}^H \boldsymbol{M}\boldsymbol{R}_{xx} \right) \boldsymbol{H}^H \left(\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^H + \boldsymbol{R}_{vv} \right)^{-1}$
- All filters first equalize the channel with $\boldsymbol{R}_{xx}\boldsymbol{H}^H \left(\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^H + \boldsymbol{R}_{vv} \right)^{-1}$, then reshape the IR with either $\tilde{\boldsymbol{b}}^H$ or $\left(\sigma_x^2 \boldsymbol{e}_m^H + \boldsymbol{b}^H \boldsymbol{M}\boldsymbol{R}_{xx} \right)$. This decreases the ISI due to the extra degree of freedom provided by these IRs.
- There is an inherent whitening matched filter front end in an equalizer designed for the MMSE criterion:

$$\boldsymbol{w}^H = \boldsymbol{e}_m^H \boldsymbol{b}^H \left(\boldsymbol{R}_{xx}^{-1} + \boldsymbol{H}^H \boldsymbol{R}_{vv}^{-1} \boldsymbol{H} \right)^{-1} \underbrace{\boldsymbol{H}^H \boldsymbol{R}_{vv}^{-1}}_{\substack{whitening \\ matched\ filter}}$$

- Design of the feedback filter can be application specific.

References

[1]: Simon Haykin, Adaptive Filter Theory, Prentice Hall, 2001,

[2]: Al-Dhahir, “FIR Channel-Shortening Equalizers for MIMO ISI Channels”, IEEE Trans. Commun., vol. 49, Feb. 2001, pp. 213-8,

[3]: R. A. Casas, et al, DFE Tutorial, http://www.ece.osu.edu/~schniter/postscript/dfetutorial.pdf,

[4]: M. Brookes, Matrix Reference Manual, http://www.ee.ic.ac.uk/hp/staff/dmb/matrix/intro.html.